Optimization of non-coplanar beam arrangements:
Expanding our delivery space

Daniel O'Connor

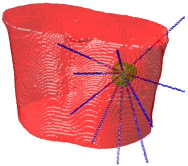
History of sparsity

Beam angle selection via group sparsity

Proximal algorithms and FISTA

Fraction-variant beam orientation optimization

Simultaneous beam angle and spot selection for IMPT



History of sparsity in signal processing

Classic problem: sparse approximation

- "Overcomplete basis": $a_1, \ldots, a_n \in \mathbb{R}^m, \quad n \gg m$
- Goal: Represent a vector $b \in \mathbb{R}^m$ as a linear combination of vectors a_i

$$b \approx \sum_i x_i a_i$$

  with most coefficients x_i equal to 0
- In matrix notation, we want

$$b \approx Ax$$

Key point: We want the vector x to be sparse

History of sparsity: Orthogonal matching pursuit

Goal:

- $F(x) = \frac{1}{2}\|Ax - b\|_2^2$ should be small
- Most components of x should be 0

Early attempt: Orthogonal matching pursuit

Strategy: Activate components of x one by one (greedy algorithm)

Repeat:

1. Select component i such that

$$\frac{\partial F}{\partial x_i} = a_i^T \underbrace{(Ax - b)}_{\text{residual}}$$

   has largest magnitude

   (In other words, select a_i that is most correlated with residual $Ax - b$)

2. Minimize $\frac{1}{2}\|Ax - b\|_2^2$ allowing only selected components to be active

Many popular beam angle selection algorithms are similar in spirit to OMP

History of sparsity: Lasso

Goal:

- $F(x) = \frac{1}{2}\|Ax - b\|_2^2$ should be small
- Most components of x should be 0

A modern approach: Lasso

Strategy: Solve optimization problem with sparsity-inducing regularizer:

$$\underset{x}{\text{minimize}} \quad \frac{1}{2}\|Ax - b\|_2^2 + \underbrace{\lambda\|x\|_1^p}_{\text{promotes sparsity}}$$

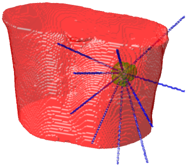
- $0 < p \leq 1$ (Lasso takes $p = 1$)
- Strategy was enabled by new optimization algorithms

Similarly, new algorithms enable the group sparsity approach to BOO

Reference: "Basis pursuit", Chen and Donoho, 1994

History of sparsity

Beam angle selection via group sparsity



Proximal algorithms and FISTA

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Beam angle selection via group sparsity

Strategy:

Solve a fluence map optimization problem with an additional **group sparsity** term that encourages most candidate beams to be inactive

$$\begin{array}{ll}\underset{x}{\text{minimize}} & \underbrace{\frac{1}{2}\|(\ell - A_0 x)_+\|_2^2}_{\text{PTV}} + \underbrace{\sum_{i=0}^{N} \frac{\alpha_i}{2}\|(A_i x - d_i)_+\|_2^2}_{\text{OARs}} + \underbrace{\sum_{b=1}^{B} w_b \|x_b\|_2^p}_{\text{group sparsity}} \\ \text{subject to} & x \geq 0\end{array}$$

- x_b is vector of beamlet intensities for bth candidate beam
- x is the concatenation of the vectors x_b
- A_0 is dose-calculation matrix for PTV
- $A_i, i = 1, \ldots, N$, is dose-calculation matrix for ith OAR

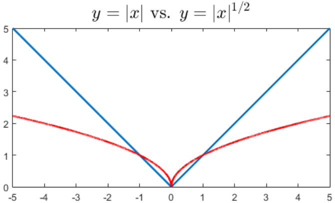
Prior work: "Beam orientation optimization for intensity modulated radiation therapy using adaptive $\ell_{2,1}$-minimization", X. Jia, C. Men, Y. Lou, and S. Jiang, 2011

Choosing the group sparsity penalty function

Previous work takes $p = 1$, solves convex problem:

- "Beam orientation optimization for intensity modulated radiation therapy using adaptive $\ell_{2,1}$-minimization", X. Jia, C. Men, Y. Lou, and S. Jiang, 2011
- "4π non-coplanar IMRT beam angle selection by convex optimization with group sparsity penalty", D. O'Connor, Y. Voronenko, D. Nguyen, W. Yin, K. Sheng, AAPM 2016

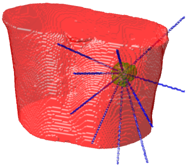
Problem: Compressed sensing informs us that $p = 1$ is only a guaranteed to be a good choice when "group restricted isometry property" is satisfied



Solution: Use a non-convex group sparsity term (we take $p = 1/2$)

History of sparsity

Beam angle selection via group sparsity



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Proximal algorithms

- **Proximal operator**

$$\text{prox}_{tg}(x) = \arg\min_u \quad g(u) + \frac{1}{2t}\|u - x\|_2^2$$

- **Proximal gradient method:** Minimizes

$$\underbrace{f(x)}_{\text{differentiable}} + \underbrace{g(x)}_{\text{easy prox-operator}}$$

via

$$x^{k+1} = \text{prox}_{tg}(x^k - t\nabla f(x^k))$$

- **FISTA:** accelerated proximal gradient method

$$y = x^k + \frac{k}{k+3}(x^k - x^{k-1})$$
$$x^{k+1} = \text{prox}_{tg}(y - t\nabla f(y))$$

FISTA with line search

Algorithm 1 FISTA with line search

Initialize x_0 and $t_0 > 0$, set $v_0 := x_0$, select $0 < r < 1$, $s > 1$
for $k = 1, 2, \ldots$ **do**
  $t := s\, t_{k-1}$
  repeat

$$\theta := \begin{cases} 1 & \text{if } k = 1 \\ \text{positive root of } t_{k-1}\theta^2 = t\theta_{k-1}^2(1-\theta) & \text{if } k > 1 \end{cases}$$

    $y := (1-\theta)x_{k-1} + \theta v_{k-1}$
    $x := \mathrm{prox}_{tg}(y - t\nabla f(y))$
    break if $f(x) \le f(y) + \langle \nabla f(y), x - y\rangle + \frac{1}{2t}\|x-y\|_2^2$
    $t := rt$
  $t_k := t$
  $\theta_k := \theta$
  $x_k := x$
  $v_k := x_{k-1} + \frac{1}{\theta_k}(x_k - x_{k-1})$
end for

References:

- "Fast first-order methods for composite convex optimization with line search", Goldfarb and Scheinberg, 2011
- Lieven Vandenberghe's UCLA 236C notes

Beam angle selection using FISTA

$$\underset{x}{\text{minimize}} \quad \frac{1}{2}\|(\ell - A_0 x)_+\|_2^2 + \sum_{i=0}^{N} \frac{\alpha_i}{2}\|(A_i x - d_i)_+\|_2^2 + \sum_{b=1}^{B} w_b \|x_b\|_2^{1/2} + I_{\geq 0}(x)$$

- Indicator penalty enforces nonnegativity:

$$I_{\geq 0}(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ \infty & \text{otherwise} \end{cases}$$

Beam angle selection using FISTA

$$\underset{x}{\text{minimize}} \quad \underbrace{\frac{1}{2}\|(\ell - A_0 x)_+\|_2^2 + \sum_{i=0}^{N} \frac{\alpha_i}{2}\|(A_i x - d_i)_+\|_2^2}_{f(x)} + \underbrace{\sum_{b=1}^{B} w_b \|x_b\|_2^{1/2} + I_{\geq 0}(x)}_{g(x)}$$

$\uparrow$
includes both group sparsity term
and indicator function

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Beam angle selection using FISTA

$$\underset{x}{\text{minimize}} \quad \underbrace{\frac{1}{2}\|(\ell - A_0x)_+\|_2^2 + \sum_{i=0}^{N} \frac{\alpha_i}{2}\|(A_ix - d_i)_+\|_2^2}_{f(x)} + \underbrace{\sum_{b=1}^{B} w_b\|x_b\|_2^{1/2} + I_{\geq 0}(x)}_{g(x)}$$

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- Indicator penalty enforces nonnegativity:

$$I_{\geq 0}(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ \infty & \text{otherwise} \end{cases}$$

- Gradient of f:

$$\nabla f(x) = -A_0^T(\ell - A_0x)_+ + \sum_{i=0}^{N} \alpha_i A_i^T(A_ix - d_i)_+$$

Beam angle selection using FISTA

$$\underset{x}{\text{minimize}} \quad \underbrace{\frac{1}{2}\|(\ell - A_0 x)_+\|_2^2 + \sum_{i=0}^{N} \frac{\alpha_i}{2}\|(A_i x - d_i)_+\|_2^2}_{f(x)} + \underbrace{\sum_{b=1}^{B} w_b \|x_b\|_2^{1/2} + I_{\geq 0}(x)}_{g(x)}$$

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- Gradient of f:

$$\nabla f(x) = -A_0^T(\ell - A_0 x)_+ + \sum_{i=0}^{N} \alpha_i A_i^T (A_i x - d_i)_+$$

- Prox-operator of g:

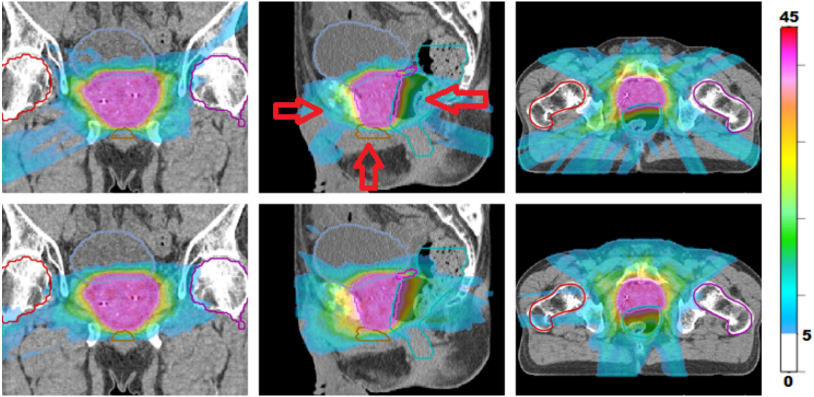
$$p_b = \text{prox}_{t w_b \|\cdot\|_2^{1/2}}(\max(x_b, 0)) = s^2 \max(x_b, 0)$$

where

$$\alpha = t w_b / \|max(x_b, 0)\|_2^{3/2}, \quad s = \begin{cases} \frac{2}{\sqrt{3}} \sin(\frac{1}{3}\arccos(\frac{3\sqrt{3}}{4}\alpha) + \frac{\pi}{2}) & \text{if } \alpha \leq \frac{2\sqrt{6}}{9} \\ 0 & \text{otherwise} \end{cases}$$

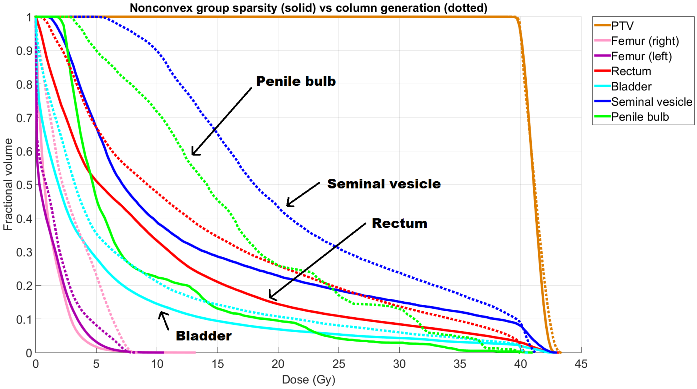
Results

Group sparsity (top row) vs. column generation (bottom row)



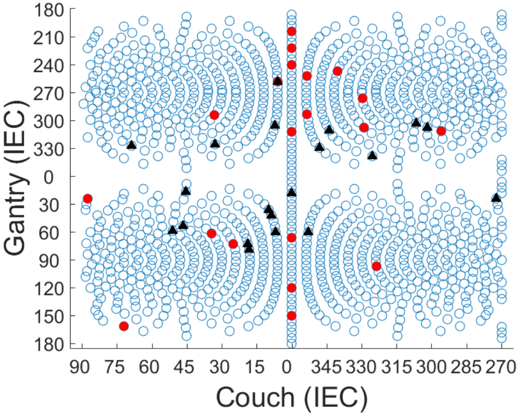
- 674 non-coplanar candidate beams
- Runtime $\approx$ 6 minutes

Results



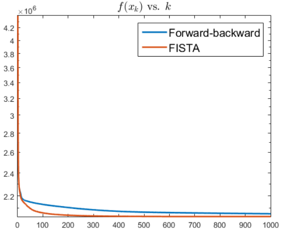
Group sparsity (solid) vs. column generation (dotted)

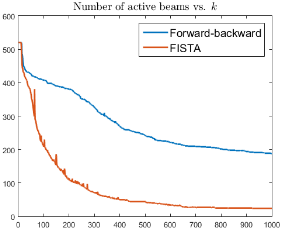
Results



Beams selected using convex (black) and non-convex (red) group sparsity terms

Proximal gradient method vs. FISTA

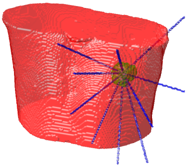




FISTA convergence rate is $O(1/k^2)$ rather than $O(1/k)$

Tricks to improve runtime:

- Prune beams as FISTA iteration progresses
- Downsample voxel grid to reduce size of matrices A_i
- Matlab trick: `A * x` $=$ `ATrans' * x`



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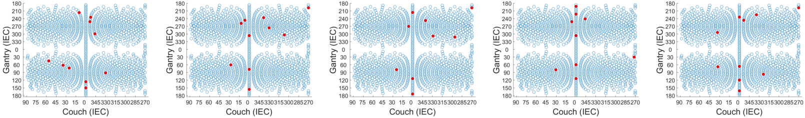
Fraction-variant beam angle selection

- Standard algorithms assume same beam angles used each treatment day
- But this seems to be an unnecessary constraint
- We should simultaneously optimize beam orientations for all fractions



Potential benefits:

- Better dosimetry
- Fewer beams per fraction $\rightarrow$ faster treatment times

Optimization formulation

$$\underset{x}{\text{minimize}} \quad \underbrace{\sum_{f=1}^{F} \frac{1}{2}\|A_0 x_f - d_0/F\|_2^2}_{\text{PTV}} + \underbrace{\sum_{i=1}^{N} \frac{\alpha_i}{2}\|(\bar{A}_i x - d_i)_+\|_2^2}_{\text{OARs}} + \underbrace{\sum_{f=1}^{F}\sum_{b=1}^{B} w_b \|x_{f,b}\|_2^p}_{\text{group sparsity}}$$

$$\text{subject to} \quad x \geq 0,$$

- F is the total number of treatment fractions
- B is the number of candidate beams *per fraction*
- A_i is dose-calculation matrix for PTV ($i = 0$) or ith OAR ($i = 1, \ldots, N$)
- $x_{f,b}$ is the fluence map for the bth candidate beam on treatment day f.
-

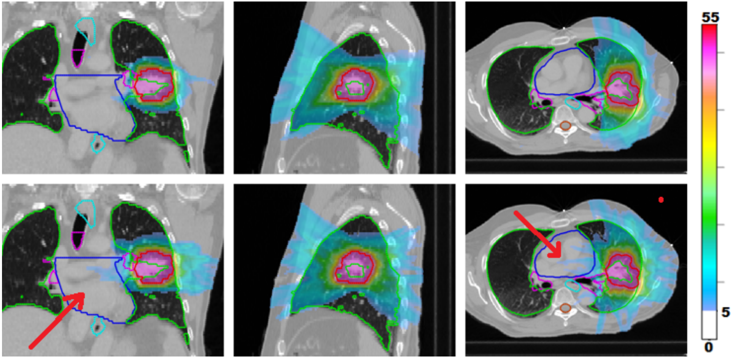
$$x_f = \begin{bmatrix} x_{f,1} \\ x_{f,2} \\ \vdots \\ x_{f,B} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_F \end{bmatrix}, \qquad \bar{A}_i = \underbrace{\begin{bmatrix} A_i & A_i & \cdots & A_i \end{bmatrix}}_{F \text{ copies of } A_i}$$

- $0 < p < 1$ (we take $p = 1/2$)

Solve using accelerated proximal gradient method (FISTA)

Lung case

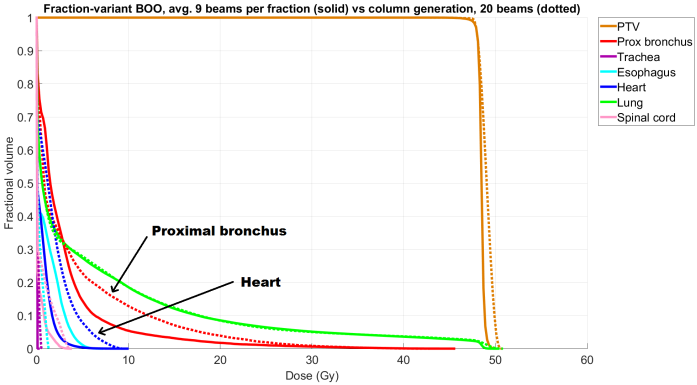
Fraction-variant plan (top row) vs. column generation plan (bottom row)



- Five fractions total, 9 beams per fraction on average
- Column generation plan used 20 beams (same beams each fraction)
- 520 candidate beams/fraction $\times$ 5 fractions $=$ 2600 candidate beams
- FISTA with line search runtime $\approx$ 65 minutes

Lung case

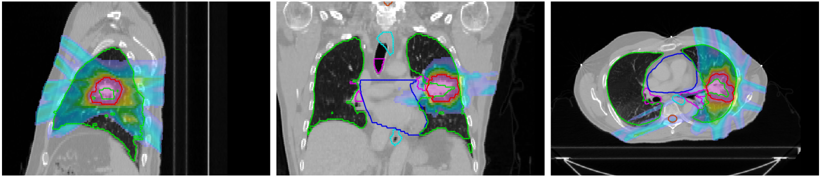
DVH for fraction-variant plan (solid) vs. column generation plan (dotted)



Fraction-variant plan: superior dosimetry, half as many beams per fraction

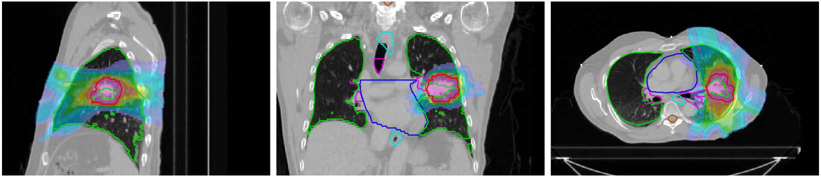
Lung case: Treatment plans for individual fractions

Fraction 1



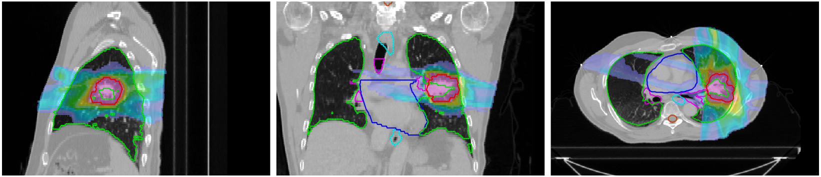
Lung case: Treatment plans for individual fractions

Fraction 2



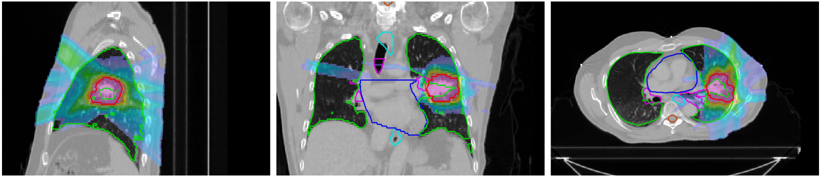
Lung case: Treatment plans for individual fractions

Fraction 3



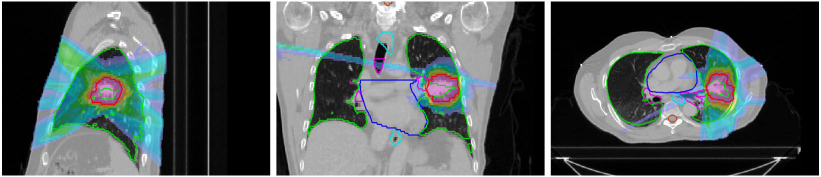
Lung case: Treatment plans for individual fractions

Fraction 4

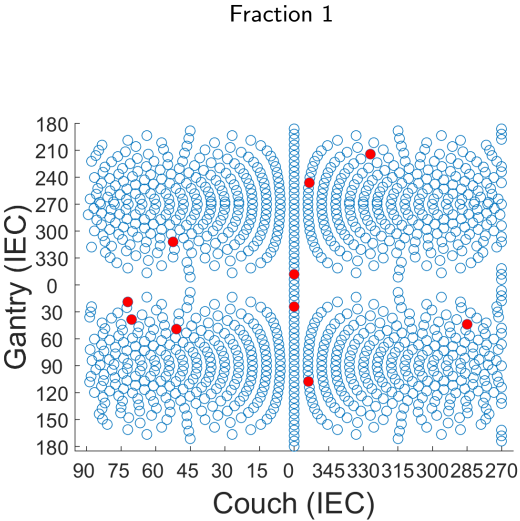


Lung case: Treatment plans for individual fractions

Fraction 5

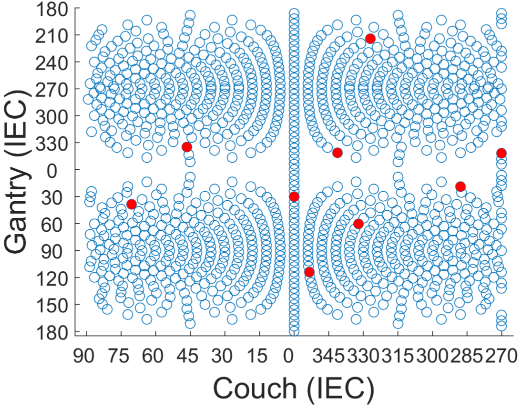


Beams selected for individual fractions

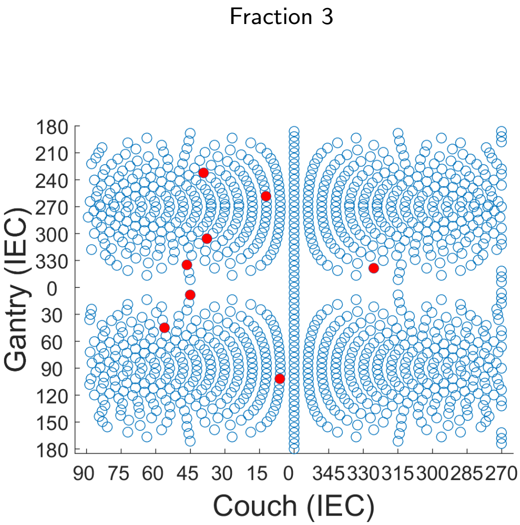


Beams selected for individual fractions

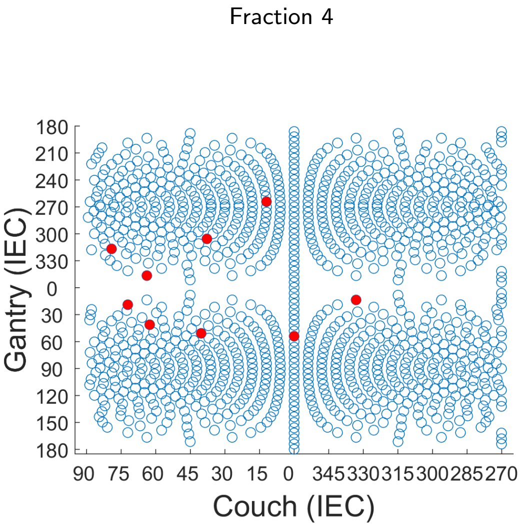
Fraction 2



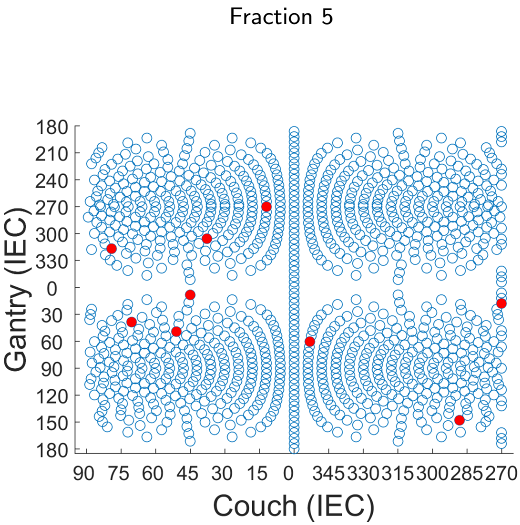
Beams selected for individual fractions



Beams selected for individual fractions

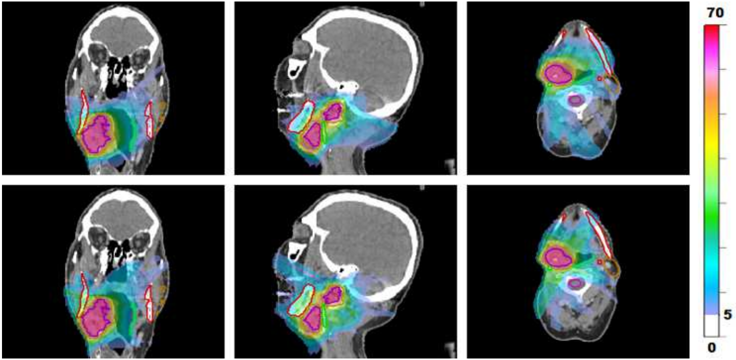


Beams selected for individual fractions



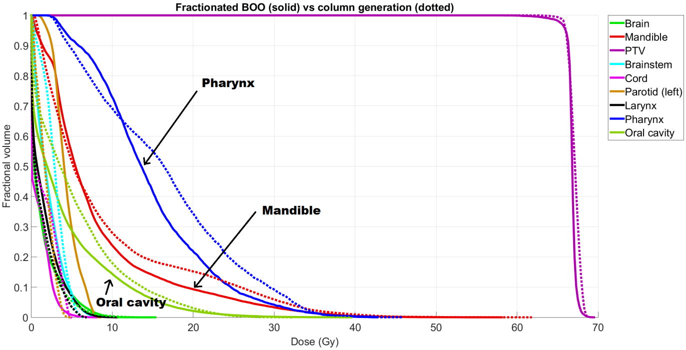
Head and neck case

Fraction-variant plan (top row) vs. column generation plan (bottom row)



- 30 fractions total, 6.5 beams per fraction on average
- Column generation plan used 20 beams (same beams each fraction)
- 811 candidate beams/fraction $\times$ 30 fractions $=$ 24,330 candidate beams
- FISTA with line search runtime $\approx$ 12 hours

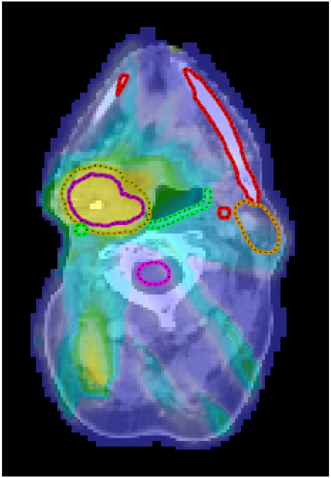
Head and neck case



Fraction-variant plan: comparable dosimetry, 1/3 as many beams per fraction

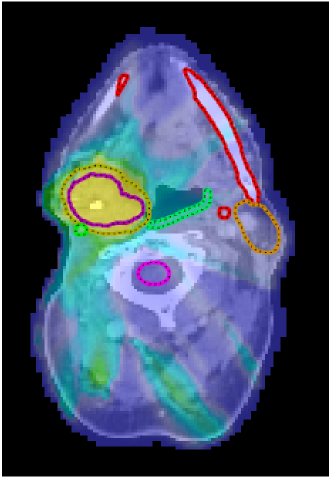
Head and neck case: Dose washes for individual fractions

Fraction 1



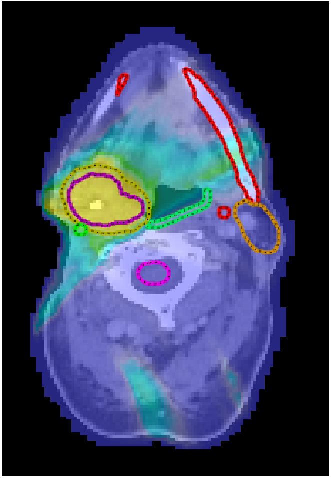
Head and neck case: Dose washes for individual fractions

Fraction 2



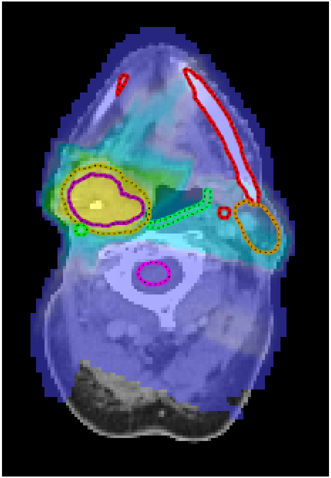
Head and neck case: Dose washes for individual fractions

Fraction 3



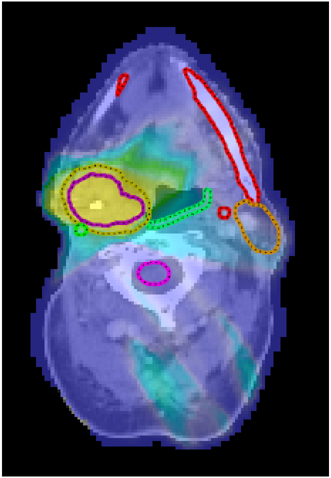
Head and neck case: Dose washes for individual fractions

Fraction 4



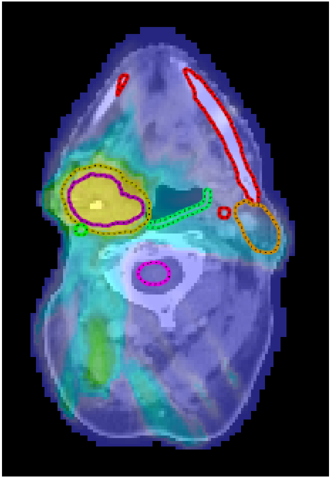
Head and neck case: Dose washes for individual fractions

Fraction 5



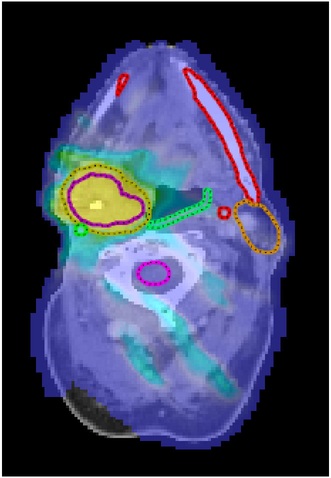
Head and neck case: Dose washes for individual fractions

Fraction 6



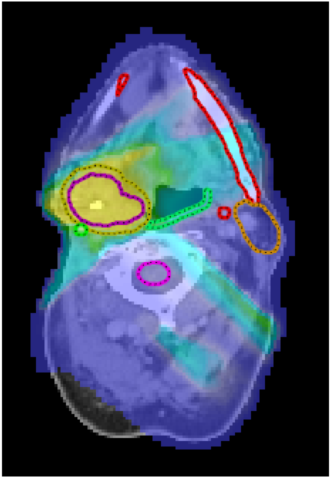
Head and neck case: Dose washes for individual fractions

Fraction 7



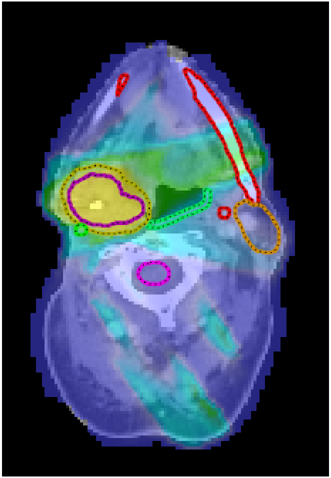
Head and neck case: Dose washes for individual fractions

Fraction 8



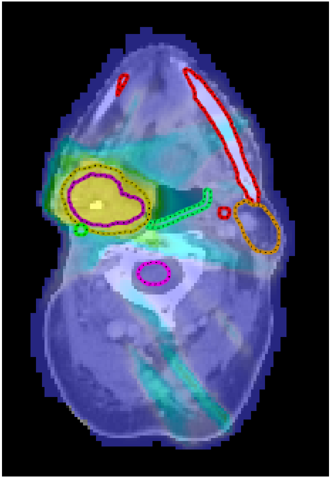
Head and neck case: Dose washes for individual fractions

Fraction 9

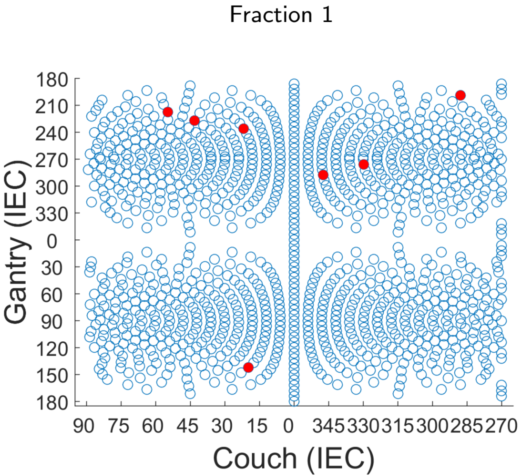


Head and neck case: Dose washes for individual fractions

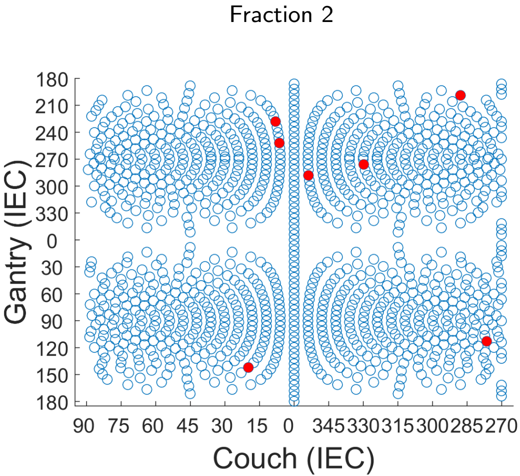
Fraction 10



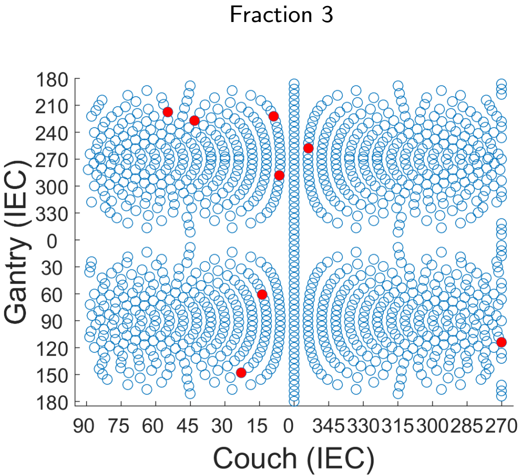
Head and neck case: Beams selected for individual fractions



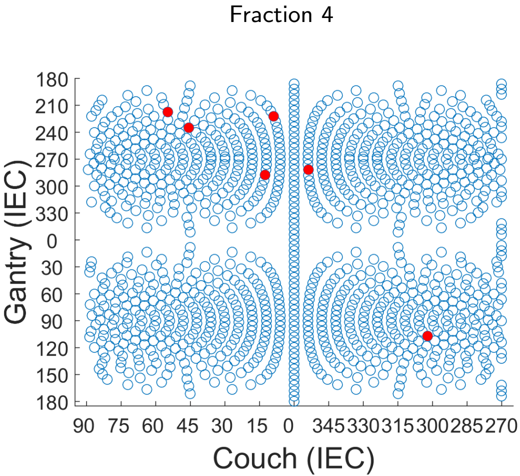
Head and neck case: Beams selected for individual fractions



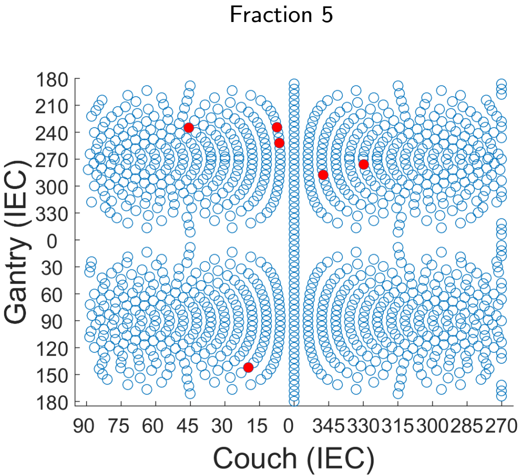
Head and neck case: Beams selected for individual fractions



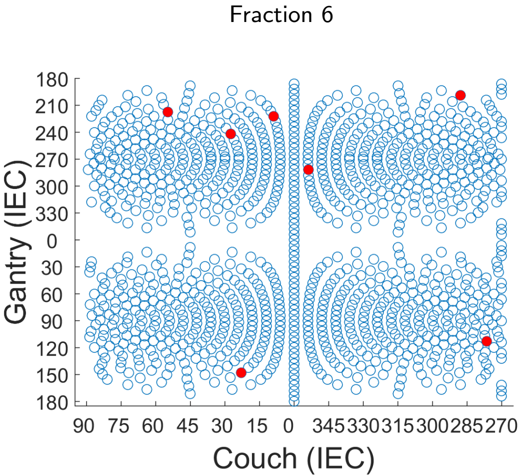
Head and neck case: Beams selected for individual fractions



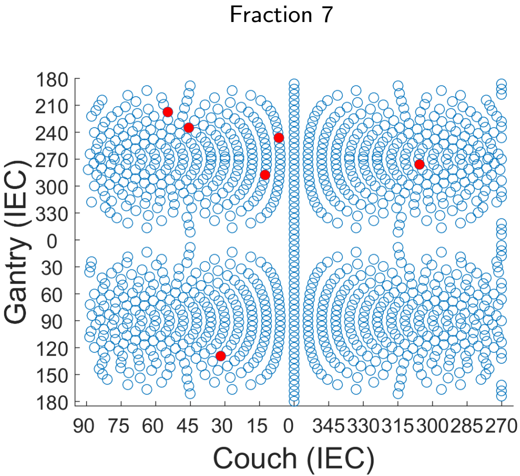
Head and neck case: Beams selected for individual fractions



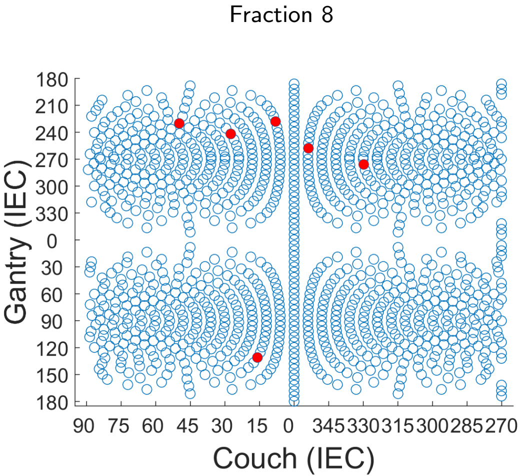
Head and neck case: Beams selected for individual fractions



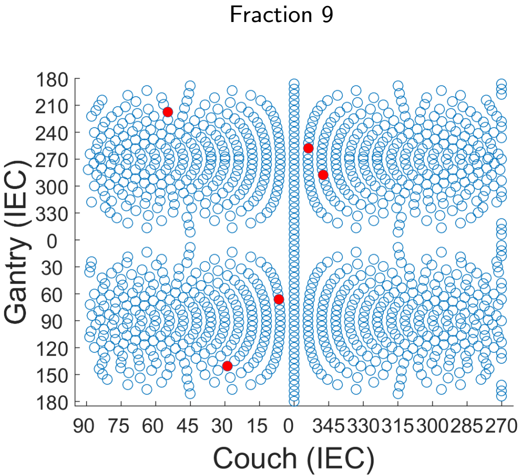
Head and neck case: Beams selected for individual fractions



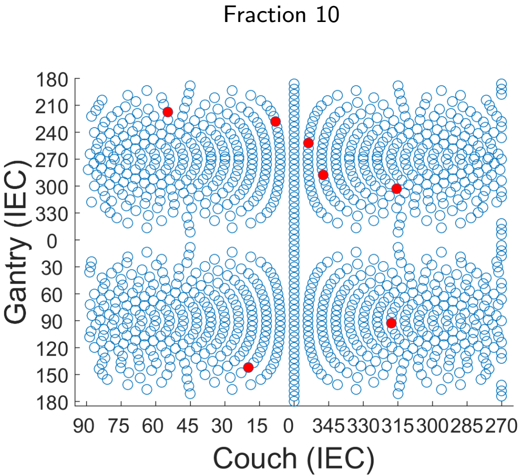
Head and neck case: Beams selected for individual fractions



Head and neck case: Beams selected for individual fractions



Head and neck case: Beams selected for individual fractions

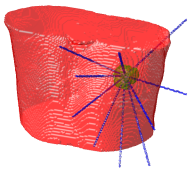


History of sparsity

Beam angle selection via group sparsity

Proximal algorithms and FISTA

Fraction-variant beam orientation optimization



Simultaneous beam angle and spot selection for IMPT

Simultaneous beam angle and spot selection for IMPT

Problem formulation:

$$\underset{x}{\text{minimize}} \quad \underbrace{\frac{1}{2}\|(\ell - A_0 x)_+\|_2^2}_{\text{PTV}} + \underbrace{\sum_{i=0}^{N} \frac{1}{2}\|(A_i x - d_i)_+\|_2^2}_{\text{OARs}} + \underbrace{\lambda \|x\|_1}_{\text{spot sparsity}} + \underbrace{\sum_{b=1}^{B} w_b \|x_b\|_2^p}_{\text{beam sparsity}}$$

$$\text{subject to} \quad x \geq 0.$$

Sparsity terms encourage x to have a hierarchical sparsity pattern:

- Sparse number of beams are active
- For each beam, sparse number of spots are active

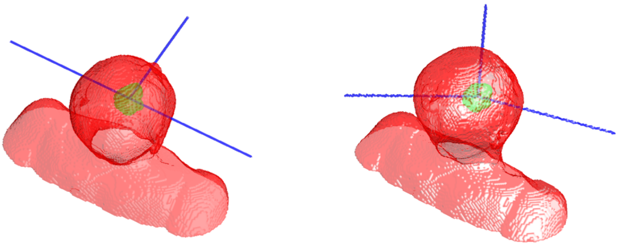
Simultaneous beam angle and spot selection for IMPT

Solution using FISTA:

$$\underset{x}{\text{minimize}} \quad \underbrace{\frac{1}{2}\|(\ell - A_0 x)_+\|_2^2 + \sum_{i=0}^{N} \frac{1}{2}\|(A_i x - d_i)_+\|_2^2}_{f(x)} + \underbrace{\lambda\|x\|_1 + \sum_{b=1}^{B} w_b \|x_b\|_2^p + I_{\geq 0}(x)}_{g(x)}$$

- Compute $\nabla f(x)$ using calculus
- Can derive explicit formula for prox-operator of g

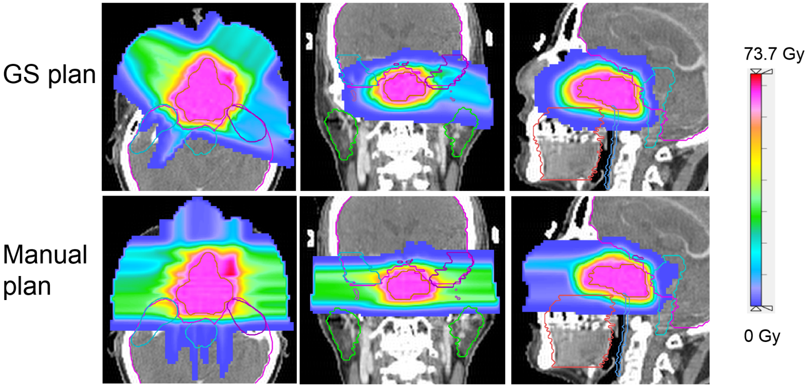
Simultaneous beam angle and spot selection for IMPT: Results



Manually selected beams (left) and group sparsity beams (right)

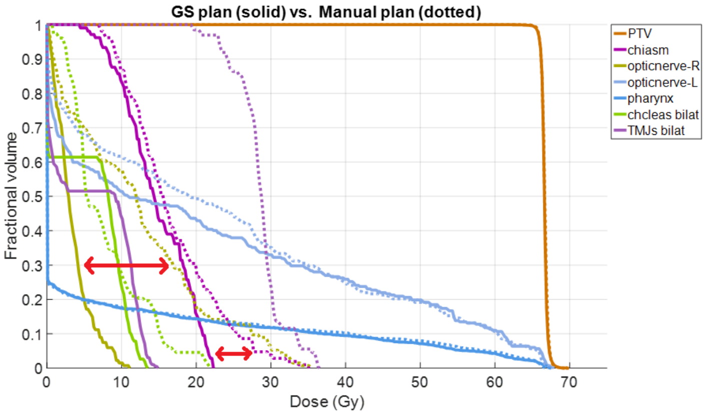
Credit: Wenbo Gu

Simultaneous beam angle and spot selection for IMPT: Results



Credit: Wenbo Gu

Simultaneous beam angle and spot selection for IMPT: Results



Credit: Wenbo Gu

Conclusions

The group sparsity approach is now efficient for non-coplanar BOO

- Improved algorithms (FISTA with line search)
- Improved hardware (more RAM)
- Potential for further improvement using GPU

Group sparsity is a useful tool in radiation treatment planning

- Non-coplanar beam orientation optimization
- Fraction-variant beam orientation optimization
- Simultaneous beam angle and spot selection for IMPT

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Thank you!

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Ke Sheng

Bonus slide: Alternative formulation

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & \sum_{f=1}^{F} \left(\underbrace{\frac{1}{2}\|A_0 x_f - d_0/F\|_2^2 + \sum_{i=1}^{N} \frac{\alpha_i}{2}\|(A_i x_f - d_i)_+\|_2^2}_{\text{controls fractional doses to PTV and OARs}} \right) \\
& + \underbrace{\sum_{i=1}^{N} \frac{\beta_i}{2}\|\bar{A}_i x\|_2^2}_{\text{controls total dose to OARs}} + \underbrace{\gamma\|Dx\|_1^{(\mu)}}_{\text{fluence map deliverability}} + \underbrace{\sum_{f=1}^{F}\sum_{b=1}^{B} w_b \|x_{f,b}\|_2^{1/2}}_{\text{group sparsity}} \\
\text{subject to} \quad & x \geq 0.
\end{aligned}
$$

Bonus slide: References

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