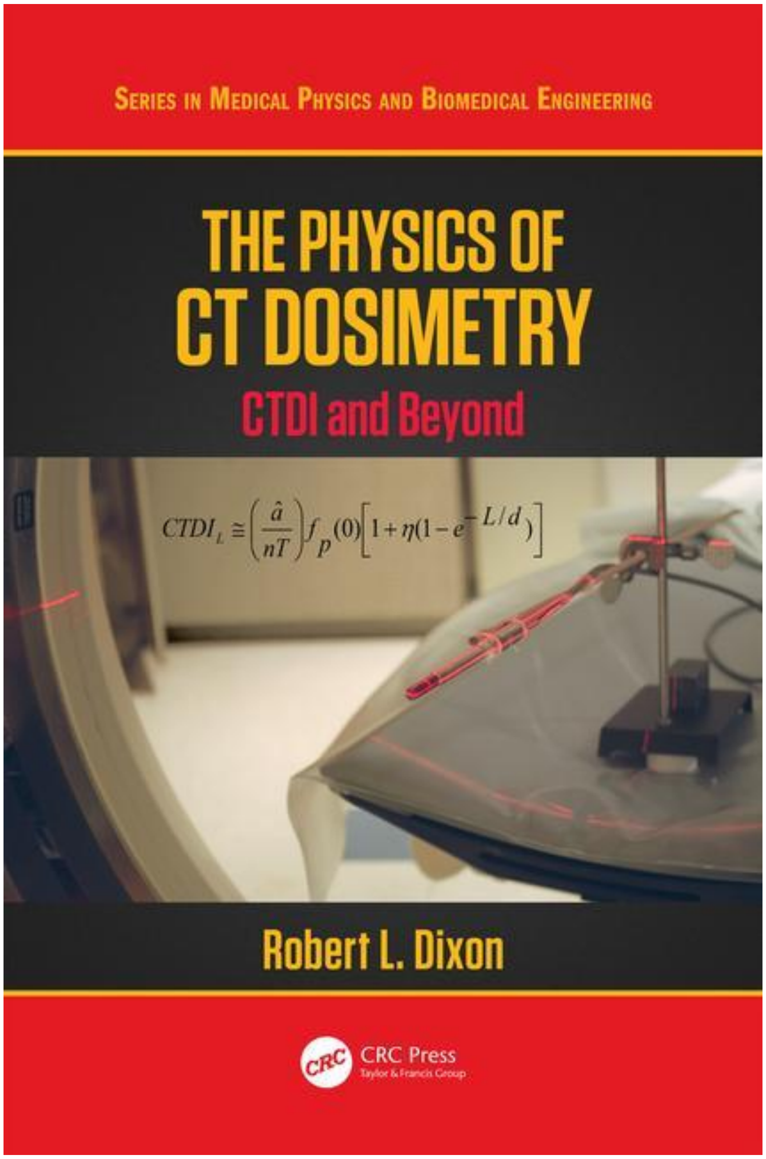
The Following Derivation Excerpted from "The Book of Bob" Chapter 7,

verse 7.3

https://www.crcpress.com/The-Physics-of-CT-Dosimetry-CTDI-and-Beyond/Dixon/p/book/9780367077594



7.3 DERIVATION OF THE THEORETICAL EQUATIONS FOR AUTOMATIC TUBE CURRENT MODULATION (TCM)

The best way to unequivocally demonstrate the nature of the CTDI-paradigm, and to illustrate how it is affected by loss of *shift-invariance* (such as TCM), is to first derive the associated equations based on a *shift-invariant* fixed tube current i_0 (fixed *mA*) technique; and then repeat the derivation using a *shift-variant* $i(z)$ TCM technique; whereby these equations (via their differences) will provide a definitive and mathematically rigorous basis for comparison of the TCM technique to the CTDI-paradigm.

7.3.1 *Shift-Invariant* Helical Technique Using Constant *mA*

Consider first a *shift-invariant* helical technique using constant *mA* in which no parameters vary with z (constant tube current i_0, pitch, aperture, etc.).

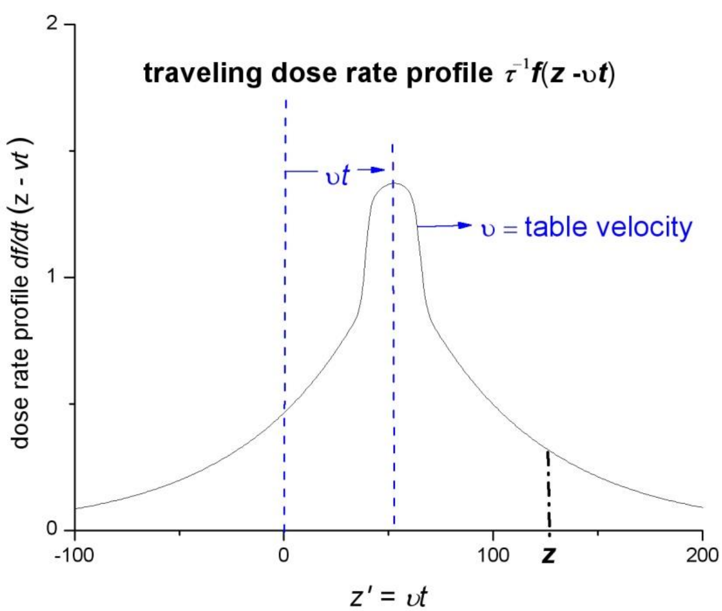


Fig. 7.1 A traveling *dose rate profile* $\dot{f}(z-\upsilon t)=\tau^{-1}f(z-\upsilon t)$ in the phantom reference frame is created when an axial dose profile $f(z)$ is translated along the phantom central axis z by table translation at velocity υ, where τ is the gantry rotation period (in sec), which has the familiar form of a traveling wave.

It is relatively simple to derive the equations of the CTDI-paradigm for the accumulated dose $D_L(z)$ on the phantom central axis in this case as shown in Chapter 2. Translation of the table and phantom at velocity υ produces a constant *dose rate* profile on the *phantom central axis* (Dixon 2003) in the form of a traveling wave $\dot{f}(z-\upsilon t)=\tau^{-1}f(z-\upsilon t)$ as depicted in Figure 7.1, where $f(z)$ is the single-rotation (axial) dose profile acquired with the phantom held stationary, and τ is the gantry rotation period (in sec); thus the dose accumulated at a fixed value of z (depicted in Figure 7.1) as the profile travels by, is given by the time-integral of $\dot{f}(z-\upsilon t)=\tau^{-1}f(z-\upsilon t)$ over the total "beam-on" time t_0, namely,

$$D_L(z)=\tau^{-1}\int_{-t_0/2}^{t_0/2} f(z-\upsilon t)dt=\frac{1}{b}\int_{-L/2}^{L/2} f(z-z')dz'=\frac{1}{b}f(z)\otimes\Pi(z/L) \qquad (7.1)$$

the conversion from the temporal to the spatial domain having been made using $z'=\upsilon t$, scan length $L=\upsilon t_0$, and a table advance per rotation $b=\upsilon\tau$, resulting in the convolution equation [Eq. (7.1)] describing the total dose $D_L(z)$ accumulated at any given z-value during the complete scan. Note that the aperture setting (and *primary beam fwhm*) for the dose profile shown Figure 7.1 is a = 26 mm, however, its wide scatter tails shown contribute to the dose at point z long before (and long after) the narrow primary beam has passed by, such that the *primary beam contribution* to $D_L(0)$ or to $CTDI_L$ is only a small fraction of the total dose. The accumulated dose at the center of the scan length $(-L/2, L/2)$ is easily obtained by setting $z = 0$ in Eq. (7.1), namely,

$$D_L(0)=\frac{1}{b}\int_{-L/2}^{L/2} f(z')dz' \qquad (7.2)$$

The close resemblance of Eq. (7.2) to the CTDI equation is obvious; indeed, $CTDI_L$ is equal to $D_L(0)$ *for one particular value of table increment* $b = nT$ (*pitch* $p = b/nT = 1$). A separate equation for $CTDI_L$ is therefore unnecessary and redundant, since $CTDI_L = pD_L(0)$. Nonetheless, we include it for later reference.

$$\mathrm{CTDI}_L = \frac{1}{nT}\int_{-L/2}^{L/2} f(z')dz' \qquad (7.3)$$

Eqs (7.1–7.3) of the CTDI-paradigm apply to *all shift-invariant scans involving table translation* (*helical or axial*) on both the peripheral axes and on the central axis, as illustrated in Chapter 2.

7.3.2 Deriving the Dose Equations for a S*hift-Variant* TCM Technique

Auto *mA* TCM is one of the simplest *shift-variant* problems, since only the *amplitude* of the traveling dose rate profile of Figure 7.1 changes with tube current $i(z)$ (and not its shape) Dixon and Boone (2013). For a variable $i(t)$, the average tube current $\langle i\rangle = \langle mA\rangle$ is taken over the entire scan time t_0, namely,

$$\langle i\rangle = \frac{1}{t_0}\int_{-t_0/2}^{t_0/2} i(t)dt = \frac{1}{L}\int_{-L/2}^{L/2} i(z')dz' \qquad (7.4)$$

where $z' = \upsilon t$, $L = \upsilon t_0$, (assuming table velocity υ and pitch are constant). Also note that the *total* $mAs = \int i(t)dt = \langle i\rangle t_0$, where t_0 is the total "beam-on" time. In order to explicitly exhibit the effect of the variable tube current $i(t)$, it can be factored out from the traveling dose rate profile $\dot{f}(z-\upsilon t, t) = i(t)\hat{f}(z-\upsilon t)$, exhibiting an explicit dependence shown as the product of $i(t)$ with a current-independent shape function $\hat{f}(z)$, *namely the axial dose profile per unit mA.*

Additionally, since the traveling dose rate function at constant current $\dot{f}(z-\upsilon t) = \tau^{-1} f(z-\upsilon t)$ implicitly contains a constant current value i_0 as noted, the aforementioned dose rate profile is modified to the equivalent form,

$$\dot{f}(z-\upsilon t,t)=i(\upsilon t)\tau^{-1}\hat{f}(z-\upsilon t)=[i(\upsilon t)/i_0]\tau^{-1}f(z-\upsilon t)) \qquad (7.5)$$

these profiles being related by $f(z-z')=i_0\hat{f}(z-z')$ where $z'=\upsilon t$ as before, and $(i_0\tau)$ is the *mAs* per rotation.

Integrating the dose rate profile in Eq. (7.5) over the total scan time t_0, and converting to the spatial domain using $z'=\upsilon t$, $L=\upsilon t_0$, as before:

The modulated tube current (*TCM*) *accumulated dose* $\tilde{D}_L(z)$ is given by,

$$\tilde{D}_L(z)=\frac{1}{b}\int_{-L/2}^{L/2} i(z')\hat{f}(z-z')dz'=\frac{1}{b}\int_{-L/2}^{L/2}\left[\frac{i(z')}{i_0}\right]f(z-z')dz'=\frac{1}{b}\hat{f}(z)\otimes\left[i(z)\Pi(z/L)\right] \qquad (7.6a)$$

The accumulated dose for a constant tube current i_0 then follows from Eq. (7.6a),

$$D_L(z)=\frac{1}{b}i_0\int_{-L/2}^{L/2}\hat{f}(z-z')dz'=\frac{1}{b}\int_{-L/2}^{L/2}f(z-z')dz'==\frac{1}{b}f(z)\otimes\Pi(z/L) \qquad (7.6b)$$

where Eq. (7.6b) is identical to Eq. (7.1) for $D_L(z)$ in the CTDI-paradigm (likewise derived under *shift-invariant* conditions) at a constant current i_0, where i_0 is implicitly imbedded in $f(z)$. It will be useful later (for comparison purposes) to set $i_0=\langle i\rangle$ of Eq. (7.4).

- These equations (7.6a and 7.6b) are seen to be quite different – in the case of variable $i(z)$ one must convolve $\hat{f}(z)$ with the $i(z)$ function, or equivalently convolve $f(z)$ with the relative $i(z)$ function $[i(z)/i_0]$. The local TCM dose $\tilde{D}_L(z)$ at z is not proportional to the local $mA(z)=i(z)$ as is often erroneously stated, but rather depends on the current $i(z')$ over all z' locations due to scattered radiation which dominates the dose in CT.
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- **That is, the physical interpretation of the product (integrand)** $i(z')\hat{f}(z-z')$ is that $i(z')$ determines the peak height of the profile at $z' = \upsilon t$, and $i(z')\hat{f}(z-z')$ represents the magnitude of the scatter tail at position *z* from that *z*′-profile, the contributions of which must be summed (integrated) *over every profile location z*′.
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Therefore, the accumulated dose in CT at a given location *z* (including $CTDI_{vol}$ at location *z* = 0) at a given time *t* depends on *mA past and mA yet to come*, and the scanner-reported "$CTDI_{vol}$ per slice" or $CTDI_{vol}(z)$ does not (and cannot) represent a dose, but merely represents the relative $i(z) \equiv mA(z)$. In fact, the dose $D_L(z)$ *even at constant mA is not constant* due to variations in such mutual profile scatter and peaks at z = 0 as seen in Chapter 2

Both these equations (7.6a and 7.6b) show that neither the TCM dose $\tilde{D}_L(z)$ nor the constant current $D_L(z)$ at *z* are proportional to the local current $i(z) = mA(z)$ at the same point. Therefore, *the average dose is not proportional to the average current, and likewise averaging the current is not the same as averaging the dose*. We can learn something about the physics involved from the convolution in Eq. (7.6a). First, the convolution (by its very nature) tends to smooth the tube current variations, reducing the effect of the local $i(z)$ on the accumulated dose $\tilde{D}_L(z)$ at the same point *z*, since $i(z)$ determines only the peak dose of a single profile located at *z*, but $\tilde{D}_L(z)$ depends on scatter contributions from many other profiles displaced relative to *z*. Physically, the wide scatter function $f_s(z)$ (Chapter 6) is primarily responsible for such smoothing, and scatter *cross-talk* between profiles reduces the influence of the local $i(z)$ on the dose $\tilde{D}_L(z)$ at that same point *z*.