Iterative Reconstruction Methods in Computed Tomography

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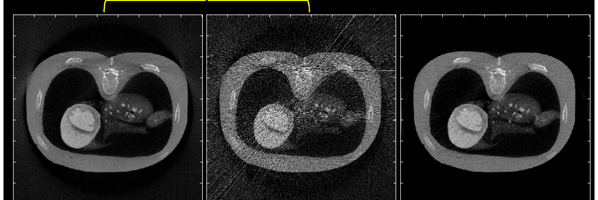
Power of Iterative Reconstruction

FBP reconstruction

Iterative Reconstruction



0.8 mAs/frame

0.1 mAs/frame

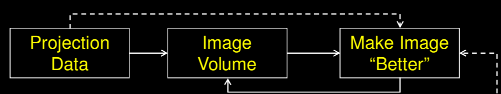
(80 kVp, 360 projections)

Learning Objectives

- Fundamentals of iterative methods
  - Approach to forming an iterative algorithm
  - Identify particular classes of methods
- Advantages of iterative approaches
  - Intuition behind what these algorithms do
  - Flexibility of these techniques
- Images produced by iterative methods
  - Differences from traditional reconstruction
  - Image properties associated with iterative reconstruction

Iterative Reconstruction

- What is iterative reconstruction?



- How can we make the image better?
  - Get a better match to the data
    - Requires a data model
  - Enforce desirable image properties
    - Encourage smoothness, edges, etc.
  - Need a measure of "better"

Building an Iterative Technique

- Define the objective
  - Find the volume that best fits the data and desired image quality

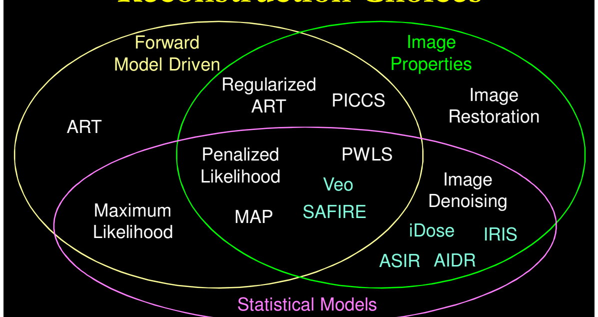
$$\text{volume} = \arg\max\{\|\text{data, model}\| \,\&\, \|\text{image properties}\|\}$$

$$\hat{\mu} = \arg\max\{\|y, \bar{y}(\mu)\| \,\&\, \|f(\mu)\|\}$$

- Devise an algorithm that solves the objective
  - Iteratively solve for $\hat{\mu}$
  - Decide when to stop iterating (and how to start)

Objective Function — Optimization Algorithm

Reconstruction Choices



*Disclaimer: The exact details of commercially available reconstruction methods are not known by the author.

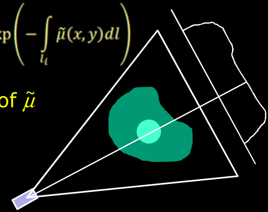
Model-based Approaches

- Transmission Tomography Forward Model
  - Projection Physics, Beer's Law

$$P(\text{photon survives}) = \exp\left(-\int_{l_i} \tilde{\mu}(x,y)dl\right)$$

$$\bar{y}_i = E[\text{number of photons}] = I_0 \exp\left(-\int_{l_i} \tilde{\mu}(x,y)dl\right)$$

- Need a Parameterization of $\tilde{\mu}$

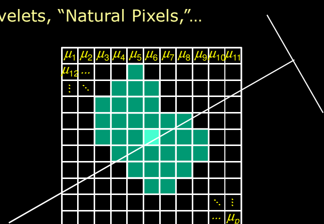


Parameterization of the Object

- Continuous-domain object
- Want finite number of parameters
- Choices of basis functions:
  - Point Samples - Bandlimited
  - Contours – Piecewise constant
  - Blobs, Wavelets, "Natural Pixels,"…
- Voxel Basis

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}$$

$$\tilde{\mu}(x,y) \cong \mathbf{B}\boldsymbol{\mu}$$



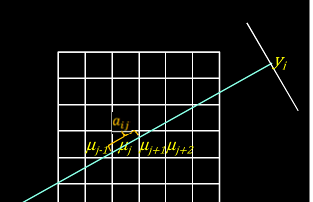
Projection

- Linear operation $\bar{y}_i = I_0 \exp\left(-\int_{l_i} \tilde{\mu}(x,y)dl\right)$
- Discrete-Discrete for parameterized problem

$$\bar{y}_i = I_0 \exp\left(-\sum_{j=1}^{p} a_{ij}\mu_j\right)$$

- System matrix

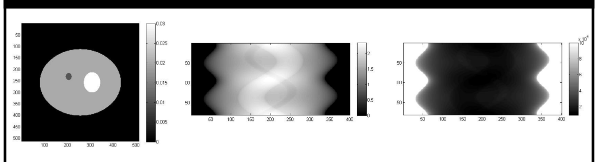
$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{ip} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{np} \end{bmatrix}$$



Forward Model

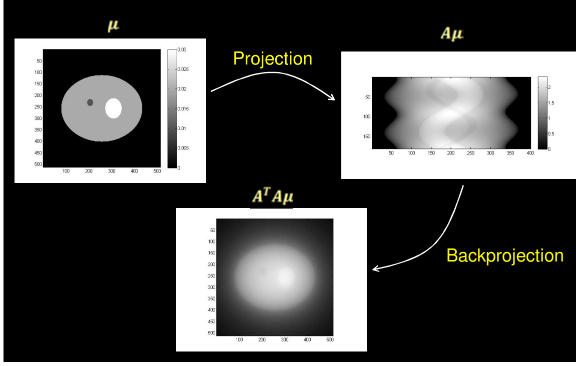
- Mean measurements as a function of parameters

$$\bar{y}(\mu) = I_0 \exp(-A\mu)$$



μ $A\mu$ $I_0 \exp(-A\mu)$

Aside: Backprojection



A Simple Model-Driven Approach

- Forward Model $\bar{y}(\mu) = I_0 \exp(-\mathbf{A}\mu)$
- Objective and Estimator

$$\hat{\mu} = \arg\min \left\| y - \bar{y}(\mu) \right\| \quad \hat{\mu} \cong \arg\min \left\| -\log\left(\frac{y}{I_0}\right) + \mathbf{A}\mu \right\|$$

$$\hat{\mu} = \arg\min \left\| \mathbf{A}\mu - l \right\|^2 = \left[\mathbf{A}^T\mathbf{A}\right]^{-1} \mathbf{A}^T l$$

Projection-Backprojection Backprojection

- For a squared-distance metric
  - Many possible algorithms
  - Can be equivalent to ART, which seeks $\mathbf{A}\mu = l$

Flexibility of Iterative Methods

$$\hat{\mu} = \arg\min \|\mathbf{A}\mu - l\|^2 = \left[\mathbf{A}^T\mathbf{A}\right]^{-1}\mathbf{A}^T l$$

- For a well-sampled, parallel beam case

$$\mathbf{A}^T\mathbf{A}x \approx \frac{1}{r} ** x \quad \rightarrow \quad \left[\mathbf{A}^T\mathbf{A}\right]^{-1} x \approx F^{-1}\{|\rho_{2D}|\} ** x$$

- For other cases

$$\left[\mathbf{A}^T\mathbf{A}\right]^{-1}\mathbf{A}^T l$$

- Iterative methods implicitly handle the geometry
  - Find the correct inversion for the specific geometry
  - Cannot overcome data nullspaces (needs complete data)

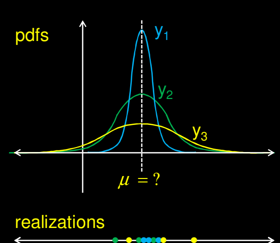
More Complete Forward Models

- More physics
  - Polyenergetic beam, energy-dependent attenuation
  - Detector effects, finite size elements, blur
  - Source effects, finite size element
  - Scattered radiation
- Noise
  - Data statistics
  - Quantum noise – x-ray photons
  - Detector Noise

Toy Problem

3 Random Variables
Different std dev ($\sigma_1, \sigma_2, \sigma_3$)

Best way to estimate μ?



Maximum Likelihood Estimation

- Find the parameter values most likely to be responsible for the observed measurements.
- Properties
  - Asymptotically unbiased and efficient under general conditions
- Likelihood Function

$$L(y;\mu) = p(y_1, y_2, \ldots, y_N | \mu_1, \mu_2, \ldots, \mu_M)$$

- Maximum Likelihood Objective Function

$$\hat{\mu} = \arg\max L(y;\mu)$$

ML Estimate for the Toy Problem

Likelihood function: $L(y,\mu) = p(y_i \mid \mu) = \prod_{i=1}^{3} p(y_i \mid \mu)$

$$p(y_i \mid \mu) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left[-\frac{1}{2\sigma_i^2}(y_i - \mu)^2\right]$$

Log-Likelihood function:

$$\log L(y,\mu) = -\frac{1}{2}\sum_{i=1}^{3} \log(2\pi\sigma_i^2) - \sum_{i=1}^{3} \frac{1}{2\sigma_i^2}(y_i - \mu)^2$$

Maximize over μ: $\hat{\mu} = \arg\max \log L(y;\mu)$

$$\frac{\partial}{\partial\mu} \log L(y,\mu) = \sum_{i=1}^{3} \frac{1}{2\sigma_i^2}(y_i - \mu)^2 \Longrightarrow \boxed{\hat{\mu} = \frac{\sum_{i=1}^{3} \frac{y_i}{\sigma_i^2}}{\sum_{i=1}^{3} \frac{1}{\sigma_i^2}}}$$

ML for Tomography

- Need a noise model
- Depends on the statistics of the measurements
- Depends on the detection process
- Common choices
  - Poisson – x-ray photon statistics

$$y_i \sim \text{Poisson}\{\bar{y}_i(\mu)\} \qquad P_{y_i} = \exp[-\bar{y}_i(\mu)] \frac{[\bar{y}_i(\mu)]^{y_i}}{y_i!}$$

  - Poisson-Gaussian mixtures – photons + readout noise

$$y_i \sim \text{Poisson}\{\bar{y}_i(\mu)\} + \text{N}\{0, \sigma_{ro}^2\}$$

  - Gaussian (nonuniform variances) – approx. many effects

$$y_i \sim \text{N}\{\bar{y}_i(\mu), \sigma_i^2\} \qquad P_{y_i} = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{1}{2\sigma_i^2}(y_i - \bar{y}_i(\mu))^2\right)$$

Poisson Likelihood Function

- Marginal Likelihoods

$$p(y_i|\boldsymbol{\mu}) = \exp[-\bar{y}_i(\boldsymbol{\mu})]\frac{[\bar{y}_i(\boldsymbol{\mu})]^{y_i}}{y_i!}$$

- Likelihood

$$L(\mathbf{y};\boldsymbol{\mu}) = p(\mathbf{y}|\boldsymbol{\mu}) = \prod_{i=1}^{N} p(y_i|\boldsymbol{\mu}) = \prod_{i=1}^{N} \exp[-\bar{y}_i(\boldsymbol{\mu})]\frac{[\bar{y}_i(\boldsymbol{\mu})]^{y_i}}{y_i!}$$

- Log-Likelihood

$$\log L(\mathbf{y};\boldsymbol{\mu}) = \sum_{i=1}^{N} y_i \log \bar{y}_i(\boldsymbol{\mu}) - \bar{y}_i(\boldsymbol{\mu}) - \cancel{\log y_i!}$$

- Objective Function

$$\hat{\mu} = \arg\max \log L(y \mid \mu)$$

Gaussian Likelihood Function

- Marginal Likelihoods

$$l_i = -\log\left(\frac{y_i}{I_0}\right) \qquad p(l_i \mid \mu) = \frac{1}{\sqrt{2\pi\sigma_i^2}}\exp\left[-\frac{1}{2\sigma_i^2}\left(l_i - [\mathbf{A}\mu]_i\right)^2\right]$$

- Likelihood

$$L(l \mid \mu) = p(l \mid \mu) = \prod_{i=1}^{N} p(l_i \mid \mu) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_i^2}}\exp\left[-\frac{1}{2\sigma_i^2}\left(l_i - [\mathbf{A}\mu]_i\right)^2\right]$$

- Log-Likelihood

$$\log L(l \mid \mu) \cong \sum_{i=1}^{N} -\frac{1}{2\sigma_i^2}\left(l_i - [\mathbf{A}\mu]_i\right)^2$$

- Objective Function and Estimator

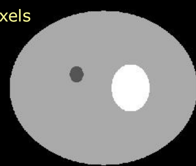
$$\hat{\mu} = \arg\max \log L(l \mid \mu) = \left[\mathbf{A}^T\mathbf{D}\left[\frac{1}{\sigma_i^2}\right]\mathbf{A}\right]^{-1}\mathbf{A}^T\mathbf{D}\left[\frac{1}{\sigma_i^2}\right]l$$

Iterative Algorithms

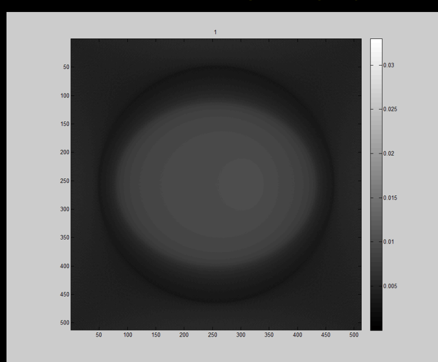
- Plethora of iterative approaches
  - Expectation-Maximization – General Purpose Methodology
  - Gradient-based Methods – Coordinate Ascent/Descent
  - Optimization Transfer – Paraboloidal Surrogates
  - Ordered-Subsets methods
- Properties of iterative algorithms
  - Monotonicity
  - True convergence
  - Speed
  - Complexity

Statistical Reconstruction Example

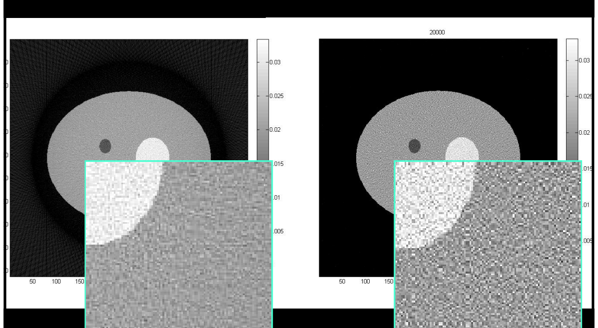
- Test Case
  - Single slice x-ray transmission problem
  - 512 x 512 0.3 mm volume
  - 400 detector bins over 180 angles (360 degrees)
  - Poisson noise: 1e5 counts per 0.5 mm detector element
  - SO: 380 mm, DO: 220 mm
- Reconstruction
  - Voxel basis: 512 x 512 0.3 mm voxels
  - Maximum-likelihood objective
  - EM-type algorithm
  - Initial image – constant value
  - Lots of iterations



ML-EM Iterations



FBP vs ML-EM Comparison



Enforcing Desirable Properties

- FBP
  - Filter designs – cutoff frequencies
- Iterative methods
  - Modify the objective to penalize "bad" images
  - Discourage noise
  - Preserve desirable image features
  - Other prior knowledge
- Image-domain Denoising

$$\hat{\mu} = \arg\max F(\mu) - \beta R(\mu)$$

- Model-based Reconstruction

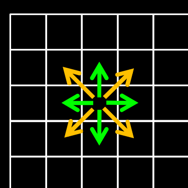
$$\hat{\mu} = \arg\max F(y, \mu) - \beta R(\mu)$$

Local Control of Image Properties

- Pairwise Penalty
  - Penalize the difference between neighboring voxels

$$R(\mu) = \sum_j \sum_{k \in N} w_{jk} \psi(\mu_j - \mu_k)$$

$$k_{1st} = \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} \quad k_{2nd} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & -1 & \frac{-1}{\sqrt{2}} \\ -1 & 4 + \frac{4}{\sqrt{2}} & -1 \\ \frac{-1}{\sqrt{2}} & -1 & \frac{-1}{\sqrt{2}} \end{bmatrix}$$



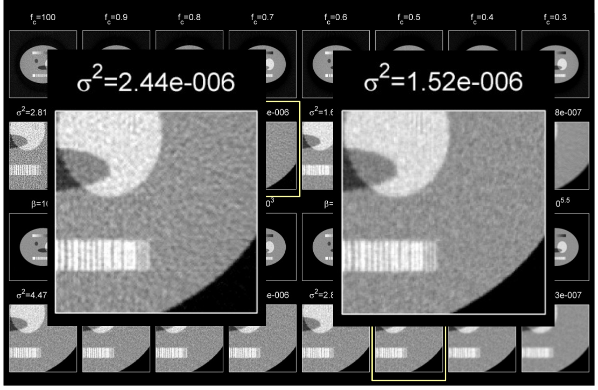
- Quadratic Penalty

$$R(\mu) = \sum_j \sum_{k \in N} w_{jk} (\mu_j - \mu_k)^2$$

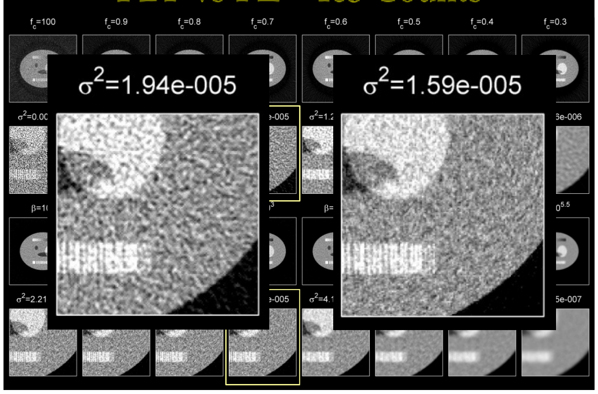
Penalized-Likelihood Example

- Forward Model
  - Fan-beam Transmission Tomography
  - 512 x 512 0.3 mm volume
  - 400 detector bins over 180 angles (360 degrees)
  - Poisson: {1e4,1e3} counts per 0.5 mm detector element
  - SO: 380 mm, DO: 220 mm
- Objective Function
  - Poisson Likelihood
  - Quadratic, first-order Penalty
- Algorithm
  - Separable Paraboloidal Surrogates
  - 400 iterations – well-converged

FBP vs PL– 1e4 Counts



FBP vs PL – 1e3 Counts



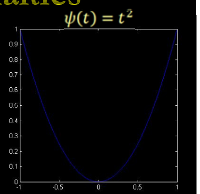
Other Penalties/Energy Functions

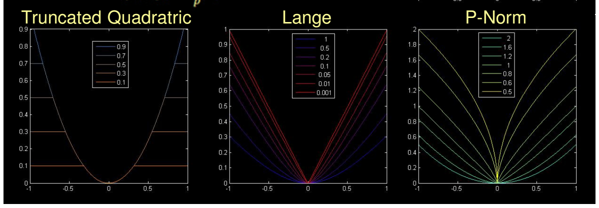
$$R(\boldsymbol{\mu}) = \sum_j \sum_{k \in N} w_{jk} \psi(\mu_j - \mu_k)$$

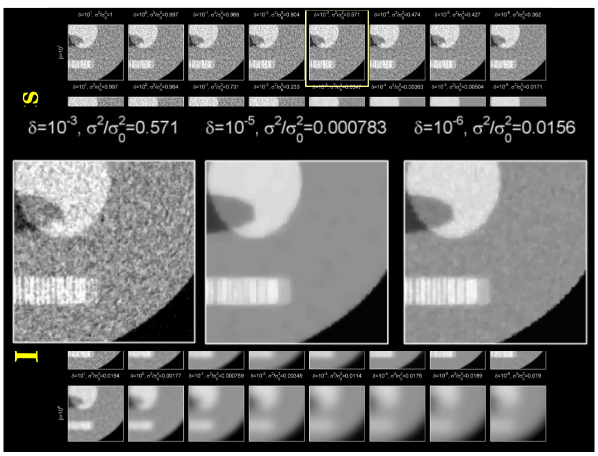
- Quadratic penalty
  - Tends to enforce smoothness throughout image
  - Increasingly penalizes larger pixel differences
- Non-quadratic penalties
  - Attempt to preserve edges in the image
  - Once pixel differences become large allow for decreased penalty (perhaps a relative decrease)
- Flexibility: Even more penalties
  - Wavelets and other bases, non-local means, etc.

Nonquadratic Penalties

- Choices
  - Truncated Quadratic
    $\psi(t;\delta) = \lfloor t^2 \rfloor_\delta$
  - Lange Penalty
    $\psi(t;\delta) = \delta\left|\frac{t}{\delta}\right| - \log\left[1 + \left|\frac{t}{\delta}\right|\right]$
  - P-Norm
    $\psi(t;p) = \frac{1}{p}|t|^p$

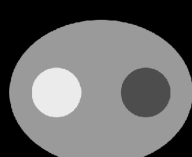






Closer Look at Image Properties

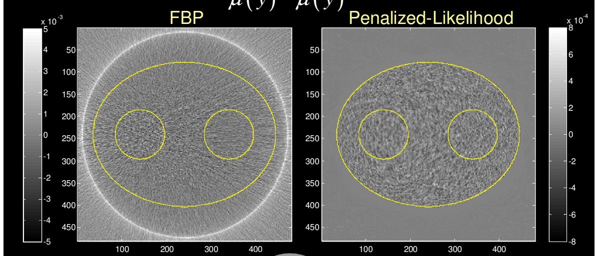
- Test Case
  - Single slice x-ray transmission problem
  - 480 x 480 0.8 mm volume
  - 1000 detector bins over 360 angles (360 degrees)
  - Poisson noise: 1e5 counts / 0.76 mm detector element
  - SO: 600 mm, DO: 600 mm



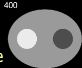
- Reconstruction
  - Penalized-likelihood objective
  - Shift-Invariant Quadratic Penalty
  - Separable paraboloidal surrogates
  - 200 iterations
  - Voxel basis: 480 x 480 0.8 mm voxels

FBP vs PL, Noise Properties

$$\hat{\mu}(y)-\hat{\mu}(\bar{y})$$



- Noise in FBP
  - Shift-variant variance
  - Shift-variant covariance



- Noise in Quadratic PL
  - Relatively shift-invariant variance (in object)
  - Shift-variant covariance

FBP vs PL, Resolution Properties

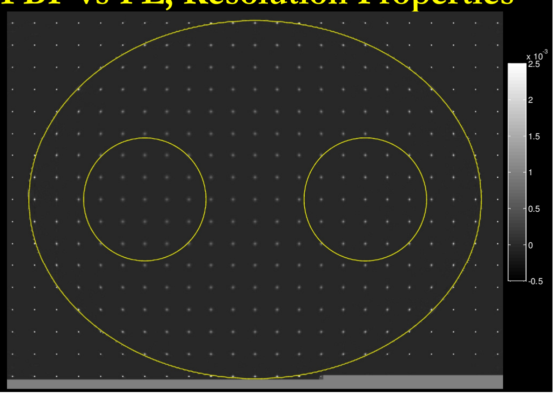


Image Properties

- Filtered-Backprojection
  - Largely shift-invariant spatial resolution
  - Shift-variant, object-dependent noise
- Uniform Quadratic Penalized Likelihood
  - Shift-variant, object-dependent spatial resolution
  - Shift-variant, object-dependent noise
- Edge-preserving Penalty Methods
  - Shift-variant, object-dependent spatial resolution
  - Shift-variant, object-dependent noise
  - Noise-resolution properties may not even be locally smooth

Learning Objectives I

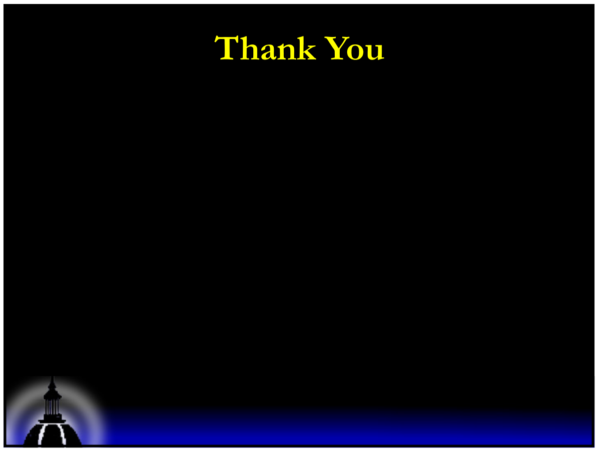
- Fundamentals of iterative methods
  - Approach to forming an iterative algorithm
    - Forward Model
    - Objective Function
    - Optimization Algorithm
  - Identify particular classes of methods
    - Model-based vs. Image Denoising approaches
    - Statistical vs. Nonstatistical approaches
    - Kinds of regularization

Learning Objectives II

- Advantages of iterative approaches
  - Intuition behind what these algorithms do
    - Fitting reconstruction to observations
    - Data weighting by information content
    - Importance of regularization
  - Flexibility of these techniques
    - Arbitrary geometries
    - Sophisticated modeling of physics
    - General incorporation of desired image properties through regularization

Learning Objectives III

- Images produced by iterative methods
  - Differences from traditional reconstruction
    - Regularization is key
    - Image properties are tied to statistical weighting
    - Can depend on algorithm when iterative solution has not yet converged
  - Image properties associated with iterative reconstruction
    - Highly dependent on regularization
    - Can be more shift-variant (esp. edge-preservation)
    - Different noise, texture


Thank You