Iterative Reconstruction Methods in Computed Tomography

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Power of Iterative Reconstruction

FBP reconstruction

Iterative Reconstruction

450 mAs
360 angles/360°

10 mAs
100 angles/200°
Learning Objectives

- Fundamentals of iterative methods
  - Approach to forming an iterative algorithm
  - Identify particular classes of methods

- Advantages of iterative approaches
  - Intuition behind what these algorithms do
  - Flexibility of these techniques

- Images produced by iterative methods
  - Differences from traditional reconstruction
  - Image properties associated with iterative reconstruction

Iterative Reconstruction

- What is iterative reconstruction?

  - Projection Data
  - Image Volume
  - Make Image “Better”

- How can we make the image better?
  - Get a better match to the data
    - Requires a data model
  - Enforce desirable image properties
    - Encourage smoothness, edges, etc.
  - Need a measure of “better”
Building an Iterative Technique

- Define the objective
  - Find the volume that best fits the data and desired image quality
    \[
    \text{volume} = \arg \max \left\{ \| \text{data, model} \| \& \| \text{image properties} \| \right\}
    \]
    \[
    \hat{\mu} = \arg \max \left\{ \| y, \tilde{y}(\mu) \| \& \| f(\mu) \| \right\}
    \]

- Devise an algorithm that solves the objective
  - Iteratively solve for \( \hat{\mu} \)
  - Decide when to stop iterating (and how to start)

Objective Function  Optimization Algorithm

Reconstruction Choices

*Disclaimer: The exact details of commercially available reconstruction methods are not known by the author.*
Model-based Approaches

- Transmission Tomography Forward Model
  - Projection Physics, Beer’s Law
    \[ I'(\text{photon survives}) = \exp\left( - \int_{l_i} \tilde{\mu}(x,y) dl \right) \]
    \[ \bar{y}_i = E[\text{number of photons}] = I_0 \exp\left( - \int_{l_i} \tilde{\mu}(x,y) dl \right) \]
  - Need a Parameterization of \( \tilde{\mu} \)

Parameterization of the Object

- Continuous-domain object
- Want finite number of parameters
- Choices of basis functions:
  - Point Samples - Bandlimited
  - Contours – Piecewise constant
  - Blobs, Wavelets, “Natural Pixels,”...
- Voxel Basis

\[ \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_P \end{bmatrix} \]

\[ \tilde{\mu}(x,y) \cong b\mu \]
**Projection**

- Linear operation: \( \bar{y}_i = l_0 \exp \left( - \int \mu(x, y) dl \right) \)
- Discrete-Discrete for parameterized problem: \( y_i = l_0 \exp \left( - \sum_{j=1}^{p} a_{ij} \mu_j \right) \)
- System matrix:

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1p} \\
    a_{21} & a_{22} & \cdots & a_{2p} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{np}
\end{bmatrix}
\]

**Forward Model**

- Mean measurements as a function of parameters: \( \bar{y}(\mu) = l_0 \exp(-A\mu) \)
Aside: Backprojection

A Simple Model-Driven Approach

- **Forward Model** \( \bar{y}(\mu) = I_0 \exp(-A\mu) \)
- **Objective and Estimator**
  \[ \hat{\mu} = \arg\min \|y - \bar{y}(\mu)\| \quad \hat{\mu} \approx \arg\min \left\| -\log \left( \frac{y}{I_0} \right) + A\mu \right\| \]
  \[ \hat{\mu} = \arg\min \|A\mu - l\|^2 = [A^T A]^{-1} A^T l \]

- **For a squared-distance metric**
  - Many possible algorithms
  - Can be equivalent to ART, which seeks \( A\mu = l \)
Flexibility of Iterative Methods

\[ \hat{\mu} = \arg \min \| A \mu - l \|^2 = A^T A^{-1} A^T l \]

- For a well-sampled, parallel beam case
  \[ A^T A x \approx \frac{1}{r} \quad \Rightarrow \quad A^T A^{-1} A^T l \]

- For other cases
  \[ A^T A^{-1} A^T l \]

- Iterative methods implicitly handle the geometry
  - Find the correct inversion for the specific geometry
  - Cannot overcome data nullspaces (needs complete data)

More Complete Forward Models

- More physics
  - Polyenergetic beam, energy-dependent attenuation
  - Detector effects, finite size elements, blur
  - Source effects, finite size element
  - Scattered radiation

- Noise
  - Data statistics
  - Quantum noise – x-ray photons
  - Detector Noise
A Simple Estimation Problem

3 Random Variables
Different std dev ($\sigma_1, \sigma_2, \sigma_3$)
Best way to estimate $\mu$?

Maximum Likelihood Estimation

- Find the parameter values most likely to be responsible for the observed measurements.

- Likelihood Function

$$ L(y; \mu) = p(y_1, y_2, \ldots, y_N \mid \mu_1, \mu_2, \ldots, \mu_M) $$

- Maximum Likelihood Objective Function

$$ \hat{\mu} = \text{arg max } L(y; \mu) $$
## ML for the Simple Example

Likelihood function:  
\[ L(y, \mu) = p(y_i | \mu) = \prod_{i=1}^{3} p(y_i | \mu) \]

\[ p(y_i | \mu) = \frac{1}{\sqrt{2\pi\sigma_i}} \exp\left[-\frac{1}{2\sigma_i^2}(y_i - \mu)^2\right] \]

Maximize over \( \mu \):  
\[ \hat{\mu} = \arg \max L(y; \mu) \]

\[ \frac{\partial}{\partial \mu} L(y, \mu) = 0 \implies \hat{\mu} = \frac{\sum_{i=1}^{3} \frac{y_i}{\sigma_i^2}}{\sum_{i=1}^{3} \frac{1}{\sigma_i^2}} \]

## ML for Tomography

- Need a noise model
- Depends on the statistics of the measurements
- Depends on the detection process
- Common choices
  - Poisson – x-ray photon statistics
    \[ y_i \sim \text{Poisson}[\bar{y}_i(\mu)] \quad p_{y_i} = \exp[-\bar{y}_i(\mu)]\frac{[\bar{y}_i(\mu)]^{y_i}}{y_i!} \]
  - Poisson-Gaussian mixtures – photons + readout noise
    \[ y_i \sim \text{Poisson}[\bar{y}_i(\mu)] + \mathcal{N}(0, \sigma_{\text{noi}}^2) \]
  - Gaussian (nonuniform variances) – approx. many effects
    \[ y_i \sim \mathcal{N}[\bar{y}_i(\mu), \sigma_i^2] \quad p_{y_i} = \frac{1}{2\pi\sigma_i^2} \exp\left(-\frac{1}{2\sigma_i^2}(y_i - \bar{y}_i(\mu))^2\right) \]
Gaussian Likelihood Function

- Marginal Likelihoods
  \[ l_i = -\log \left( \frac{y_i}{I_0} \right) \quad p(l_i | \mu) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp \left[ -\frac{1}{2\sigma_i^2} \left( l_i - [A\mu]_i \right)^2 \right] \]

- Likelihood
  \[ L(l | \mu) = p(l | \mu) = \prod_{i=1}^{N} p(l_i | \mu) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp \left[ -\frac{1}{2\sigma_i^2} \left( l_i - [A\mu]_i \right)^2 \right] \]

- Log-Likelihood
  \[ \log L(l | \mu) \approx \sum_{i=1}^{N} -\frac{1}{2\sigma_i^2} \left( l_i - [A\mu]_i \right)^2 \]

- Objective Function and Estimator
  \[ \hat{\mu} = \arg \max \log L(l | \mu) = \left[A^T D \left[ \frac{1}{\sigma_i^2} \right] A \right]^{-1} A^T D \left[ \frac{1}{\sigma_i^2} \right] l \]

Iterative Algorithms

- Plethora of iterative approaches
  - Expectation-Maximization – General Purpose Methodology
  - Gradient-based Methods – Coordinate Ascent/Descent
  - Optimization Transfer – Paraboloidal Surrogates
  - Ordered-Subsets methods

- Properties of iterative algorithms
  - Monotonicity
  - True convergence
  - Speed
  - Complexity
Statistical Reconstruction Example

- **Test Case**
  - Single slice x-ray transmission problem
  - 512 x 512 0.3 mm volume
  - 400 detector bins over 180 angles (360 degrees)
  - Poisson noise: 1e5 counts per 0.5 mm detector element
  - SO: 380 mm, DO: 220 mm

- **Reconstruction**
  - Voxel basis: 512 x 512 0.3 mm voxels
  - Maximum-likelihood objective
  - EM-type algorithm
  - Initial image – constant value
  - Lots of iterations

ML-EM Iterations
FBP vs ML-EM Comparison

Enforcing Desirable Properties

- FBP
  - Filter designs – cutoff frequencies

- Iterative methods
  - Modify the objective to penalize “bad” images
  - Discourage noise
  - Preserve desirable image features
  - Other prior knowledge

- Image-domain Denoising
  \[ \hat{\mu} = \arg \max F(\mu) - \beta R(\mu) \]

- Model-based Reconstruction
  \[ \hat{\mu} = \arg \max F(y, \mu) - \beta R(\mu) \]
Local Control of Image Properties

- **Pairwise Penalty**
  - Penalize the difference between neighboring voxels

\[ R(\mu) = \sum_{j} \sum_{k \in N} w_{jk} (\mu_j - \mu_k) \]

- **Quadratic Penalty**

\[ R(\mu) = \sum_{j} \sum_{k \in N} w_{jk} (\mu_j - \mu_k)^2 \]

Penalized-Likelihood Example

- **Forward Model**
  - Fan-beam Transmission Tomography
  - 512 x 512 0.3 mm volume
  - 400 detector bins over 180 angles (360 degrees)
  - Poisson: \{1e4,1e3\} counts per 0.5 mm detector element
  - SO: 380 mm, DO: 220 mm

- **Objective Function**
  - Poisson Likelihood
  - Quadratic, first-order Penalty

- **Algorithm**
  - Separable Paraboloidal Surrogates
  - 400 iterations – well-converged
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FBP vs PL – 1e4 Counts

FBP vs PL – 1e4 Counts

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FBP vs PL – 1e3 Counts

FBP vs PL – 1e3 Counts
Other Penalties/Energy Functions

\[ \mathcal{H} = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} w_{jk} \psi(\mu_j - \mu_k) \]

- **Quadratic penalty**
  - Tends to enforce smoothness throughout image
  - Increasingly penalizes larger pixel differences

- **Non-quadratic penalties**
  - Attempt to preserve edges in the image
  - Once pixel differences become large allow for decreased penalty (perhaps a relative decrease)

- **Flexibility: Even more penalties**
  - Wavelets and other bases, non-local means, etc.

### Nonquadratic Penalties

- **Choices**
  - Truncated Quadratic
    \[ \psi(t; \delta) = |t|^2 \delta \]
  - Lange Penalty
    \[ \psi(t; \delta) = \delta \left[ |t| - \log \left( 1 + \frac{|t|}{\delta} \right) \right] \]
  - P-Norm
    \[ \psi(t; p) = \frac{1}{p} |t|^p \]

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Truncated Quadratic

Lange

P-Norm
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Closer Look at Image Properties

- **Test Case**
  - Single slice x-ray transmission problem
  - 480 x 480 0.8 mm volume
  - 1000 detector bins over 360 angles (360 degrees)
  - Poisson noise: 1e5 counts / 0.76 mm detector element
  - SO: 600 mm, DO: 600 mm

- **Reconstruction**
  - Penalized-likelihood objective
  - Shift-Invariant Quadratic Penalty
  - Separable paraboloidal surrogates
  - 200 iterations
  - Voxel basis: 480 x 480 0.8 mm voxels

FBP vs PL, Noise Properties

- **Noise in FBP**
  - Shift-variant variance
  - Shift-variant covariance

- **Noise in Quadratic PL**
  - Relatively shift-invariant variance (in object)
  - Shift-variant covariance
FBP vs PL, Resolution Properties
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**PL Image Properties – Thorax**

- Filtered-Backprojection
  - Largely shift-invariant spatial resolution
  - Shift-variant, object-dependent noise

- Uniform Quadratic Penalized Likelihood
  - Shift-variant, object-dependent spatial resolution
  - Shift-variant, object-dependent noise

- Edge-preserving Penalty Methods
  - Shift-variant, object-dependent spatial resolution
  - Shift-variant, object-dependent noise
  - Noise-resolution properties may not even be locally smooth

**Image Properties**
Learning Objectives I

- Fundamentals of iterative methods
  - Approach to forming an iterative algorithm
    - Forward Model
    - Objective Function
    - Optimization Algorithm
  - Identify particular classes of methods
    - Model-based vs. Image Denoising approaches
    - Statistical vs. Nonstatistical approaches
    - Kinds of regularization

Learning Objectives II

- Advantages of iterative approaches
  - Intuition behind what these algorithms do
    - Fitting reconstruction to observations
    - Data weighting by information content
    - Importance of regularization
  - Flexibility of these techniques
    - Arbitrary geometries
    - Sophisticated modeling of physics
    - General incorporation of desired image properties through regularization
Learning Objectives III

- Images produced by iterative methods
  - Differences from traditional reconstruction
    - Regularization is key
    - Image properties are tied to statistical weighting
    - Can depend on algorithm when iterative solution has not yet converged
  - Image properties associated with iterative reconstruction
    - Highly dependent on regularization
    - Data- and object dependent
    - Shift-variant; different noise, texture than FBP

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